Energy level statistics of a critical random matrix ensemble

2003 J. Phys. A: Math. Gen. 36 3639

(http://iopscience.iop.org/0305-4470/36/12/344)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 3639–3645 PII: S0305-4470(03)52537-9

Energy level statistics of a critical random matrix ensemble

Macleans L Ndawana[1,3] **and Vladimir E Kravtsov**[2,4]

[1] Institut für Physik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany
[2] International Center for Theoretical Physics, PO Box 586, 34100 Trieste, Italy

Received 15 August 2002, in final form 6 January 2003
Published 12 March 2003
Online at stacks.iop.org/JPhysA/36/3639

Abstract
We study energy level statistics of a critical random matrix ensemble of power-law banded complex Hermitian matrices. We compute the level compressibility via the level-number variance and compare it with the analytical formula for the exactly solvable model of Moshe, Neuberger and Shapiro.

PACS numbers: 02.10.Yn, 02.60.−x, 05.45.Mt

1. Introduction

Random matrix ensembles (RME) provide a natural framework for the statistical description of quantized energy levels of complex systems. This is probably best explained by paraphrasing Dyson [1], in ordinary statistical mechanics a renunciation of knowledge about the system is made i.e., by assuming that all states of a large ensemble are equally probable. Therefore, only the overall behaviour of the system is accessible. Thus orthodox statistical mechanics is clearly inadequate for the discussion of energy level statistics of complex systems. In energy level statistics one wishes to make statements about the finer details of the energy level structure, and such statements cannot be made in terms of an ensemble of states. Random matrix theory (RMT) turns out to be a suitable tool in making such statements about the eigen-spectrum of complex systems e.g., energy levels of highly excited nuclei [2]. The underlying assumption in RMT is that one renounces exact knowledge *not* of the state but of the *nature* of the system itself. This is the fundamental difference between ordinary statistical mechanics and energy level statistics.

There are three RMEs that emerge from an analysis of real eigen-spectra of random Hermitian matrices [3]; the Gaussian orthogonal ensemble (GOE) for a system with time-reversal without spin-$\frac{1}{2}$ interactions, the Gaussian symplectic ensemble (GSE) for a system with time-reversal and spin-$\frac{1}{2}$ interaction and the Gaussian unitary ensemble (GUE) for a system without time-reversal. These ensembles (generally called Wigner–Dyson (WD)

[3] Permanent address: University of Zambia, Department of Physics, Lusaka, Zambia.
[4] Permanent address: Landau Institute for Theoretical Physics, Kosygina str. 2, 117940 Moscow, Russia.

statistics) exhibit strong energy-level correlations leading to the phenomenon of level repulsion [4] in their respective eigen-spectra. One could add the Poisson statistics as a fourth ensemble or the second statistics is the Poisson statistics (PS) which does not exhibit level repulsion in its corresponding eigen-spectra.

In general, the WD statistics describe a system of delocalized states and the PS describe a system of localized states. The transition between these two cases controlled by the strength of disorder (i.e., the Anderson model [5]) is a quantum critical phenomenon. At a critical disorder the states acquire a special property of *multifractality* [6, 7] which marks a qualitative difference as compared to truly delocalized states in a metal and truly localized states in an insulator. These critical states correspond to the *critical level statistics* (CLS) [8, 9].

There are numerous suggestions in the literature [6, 7, 10–17] describing CLS by ensembles of random matrices. Initially, there were two main requirements for the critical RME the first was that the level statistics be system-size independent and secondly, the ensemble should be able to interpolate between WD and PS [8]. Later on it was realized [6, 18] that the key property of the critical states relevant for the CLS is multifractality. The power-law banded RME suggested in [11] emerged as the best candidate for the critical RME as its level statistics are known to possess these two properties and its eigenfunction statistics are multifractal. Thus it has been suggested [6] to be the critical random matrix ensemble (CrRME). In the limit of large bandwidth this model turns out to be the best deformation of the WD-RME that generates correlations in the eigenstates but retains all the basic properties of WD theory, including the factorizability of higher spectral correlation functions and mapping onto the Calogero–Sutherland model of fictitious interacting fermions [19].

In the absence of an exact solution of CrRME various mappings on and approximate correspondences with exactly solvable models have been used in order to study the properties and CrRME. Here we mention the mapping onto the nonlinear sigma-model [11], and the approximate equivalence [6] between the CrRME and the exactly solvable RME suggested in [10, 12]. Making use of the exact solution [12] it was possible [20] to compute the level spacing distribution function (LSDF) for this model and compare it with the exact diagonalization of the 3D Anderson model for the critical value of disorder. The coincidence of results (for certain values of the parameter b which were found from the fitting of the far tails of LSDF) was amazing [20, 23]: in all three Dyson symmetry classes the deviation was not larger than between the Wigner surmise and the exact WD distribution function.

In this paper we study the level number variance in the CrRMT suggested [11]. The ultimate goal here is to compare the numerical simulations of this critical ensemble and the analytical results for the exactly solvable ensemble in [10] (which will be referred to as MNS) for all values of the control parameter b. For large $b \gg 1$ (weak multifractality) these two ensembles are known to be equivalent [6]. The first correction to the Poisson spectral statistics at small $b \ll 1$ (strong multifractality) is also the same [21, 22]. We would like to check these predictions numerically for the unitary CrRME and to see how large are the deviations between these two RME for $b \sim 1$.

The paper is organized as follows: in section 2 we give a general overview of the level-number variance (Σ_2-statistics) and the calculation of the level compressibility, in section 3 we give some numerical details and finally a summary and outlook of this work is given in section 4.

2. Σ_2-statistics and level compressibility

Our model is the power-law random banded matrix defined as an ensemble of random Hermitian $N \times N$ matrices belonging to GUE. The real and imaginary parts of the matrix

elements are independently distributed Gaussian random variables with zero mean $\langle H_{ij}\rangle = 0$ and the variance of the matrix elements is given by

$$\langle |H_{ij}|^2\rangle = \left[1 + \frac{1}{b^2}\frac{\sin^2(\pi|i-j|/N)}{(\pi/N)^2}\right]^{-1} \times \begin{cases} \frac{1}{\beta} & \text{if} \quad i = j \\ \frac{1}{2} & \text{if} \quad i \neq j \end{cases} \tag{1}$$

where b is a parameter that controls the fractal dimensions of the critical eigenfunctions and β is the symmetry parameter i.e., $\beta = 1, 2$ and 4 for the GOE, GUE and GSE respectively. This model remains critical for any arbitrary value of b.

From a physical point of view the GUE can be realized by putting a system in a strong external magnetic field i.e., provided the splitting of levels by the magnetic field is of the same order of magnitude as the average level spacing in the absence of the field.

In this paper we study the level compressibility $\chi(N, b)$ via the level-number variance $\Sigma_2(\langle n\rangle)$ of the above model. The level-number variance is a statistical quantity that provides a quantitative measure of the long-range 'rigidity' of the energy spectrum [3];

$$\Sigma_2(\langle n\rangle) = \langle n^2\rangle - \langle n\rangle^2. \tag{2}$$

The behaviour of Σ_2 is known (as a function of the mean number of levels $\langle n\rangle \gg 1$) i.e., in the delocalized and localized phases, respectively. However, in the critical regime, the level-number variance has been conjectured to be Poisson-like [18, 24];

$$\Sigma_2 \sim \begin{cases} \frac{2}{\pi^2\beta}\ln(\langle n\rangle) & \text{delocalized} \\ \chi\langle n\rangle & \text{critical} \\ \langle n\rangle & \text{localized} \end{cases} \tag{3}$$

where the level compressibility χ is another important parameter to characterize the localization–delocalization transition (LDT) and takes values $0 \leqslant \chi \leqslant 1$, being zero in the delocalized state and unity in the localized state. Formally, one computes χ as

$$\chi \approx \lim_{\langle n\rangle\to\infty}\lim_{N\to\infty}\frac{\mathrm{d}\Sigma_2(\langle n\rangle)}{\mathrm{d}\langle n\rangle}. \tag{4}$$

The limits in equation (4) do not commute and their non-commutativity is attributed to the violation of the normalization sum rule [18]. It is remarkable (see [6, 18, 25, 26] for details) that χ can be expressed in terms of the fractal or the correlation dimension D_2 and the spatial dimension d. In [18] an explicit and very simple relationship between the critical spectral and eigenfunction statistics has been suggested:

$$\chi = \frac{1}{2}\left(1 - \frac{D_2}{d}\right). \tag{5}$$

There is vast analytical and numerical evidence that this formula is correct at weak multifractality. However, its status for the case of strong multifractality (small D_2) raises much doubts [22]. We later see that whilst our numerical data are in agreement with equation (5) in the limit $b^{-1} \to 0$ they are in disagreement [21, 22] for the opposite limit. At the same time the overall agreement with the analytical result (10) for the MNS model is unexpectedly good for all values of b and the moderately large matrix sizes up to $N = 400$. We were not able to detect any systematic dependence on the size of the matrix N in this range.

In the next section we turn to the numerical details of how to compute the level compressibility.

3. Results and discussion

We performed numerical simulations of the RME described by equation (1) for values of $b^{-1} \in [0.4, 40]$. The matrices are not sparse and as such the use of fast and efficient algorithms i.e., the Lanczos algorithm is not possible. However, we calculated the spectra of $N \times N$ matrices for $N = 50$ with 10 000 realizations to $N = 400$ with 2000 realizations by exact numerical diagonalization and hence accumulating $\sim 1 \times 10^5$ eigenvalues for each value of N. The results we discuss here are for the GUE only.

The average density of states as a function of energy E is well described by

$$\rho(E) = \frac{N}{\pi^2 b} \int_0^{+\infty} \frac{\mathrm{d}t}{1 + (1/z)\exp[(\beta/2)(E^2 + t^2)]} \tag{6}$$

where $z = \exp(\pi\beta b) - 1$. The two limiting cases of equation (6) are

$$\rho(E) = \begin{cases} \frac{N}{\pi^2 b}\sqrt{2\pi b - E^2} & \text{Semi-circle if } \pi\beta b \gg 1 \\ N\sqrt{\frac{\beta}{2\pi}}\exp(-\beta E^2/2) & \text{Gaussian otherwise.} \end{cases} \tag{7}$$

All our calculations are performed at the band centre (containing $\langle n \rangle \gg 1$) where the density of states is nearly constant. The calculation of χ from Σ_2 is non-trivial as there are finite-size effects $\sim -\langle n \rangle^2/N$ in Σ_2 originating from the long tail $R(s) \sim -1/N$ of the two-level correlation function. This is circumvented by fitting Σ_2 data with a polynomial in $\langle n \rangle$ of degree m:

$$\Sigma_2(\langle n \rangle) \approx \chi_0 + \chi\langle n \rangle + \sum_{k=2}^{m} \chi_k \langle n \rangle^k. \tag{8}$$

Then we identify the coefficient of the *linear* term as χ and χ_0 is the *constant* term. Both these terms are expected to have only weak N-dependence. This is in contrast to that of the coefficients $\chi_k \propto 1/N^{k-1}$ with $k > 1$ that have a strong N-dependence which describe finite-size effects in Σ_2. By making the polynomial fit (8) we separate the finite-size effects in Σ_2 and the weak N-dependence in χ and in χ_0. This weak N-dependence in χ and in χ_0 could be of fundamental interest and needs further investigation.

In figure 1 we plot Σ_2 for $N = 400$ and $\Sigma_2(\langle n \rangle)$ is linear as a function of $\langle n \rangle$ as is expected for a critical ensemble. The lines through the data points are in agreement with the Poisson-like behaviour i.e., $\Sigma_2 \approx \chi\langle n \rangle + \chi_0$. The value of χ is in good agreement with equation (5) and equation (9) that follows from the theoretical prediction of equation (5) [13, 25] for large πb:

$$\chi = 1/4\pi b. \tag{9}$$

However, our results are in disagreement with equation (5) for small πb [21, 22]. The analytic formula for χ (MNS in figure 2) that follows from [10] for *all* values of b reads:

$$\chi = \frac{\mathrm{d}\ln(\ell)}{\mathrm{d}\ln(z)} = \frac{\mathrm{PolyLog}[-1/2, 1 - \exp(2\pi b)]}{\mathrm{PolyLog}[+1/2, 1 - \exp(2\pi b)]} \tag{10}$$

where $\ell = \oint_0^\infty \frac{1}{(1+z^{-1}\exp(p^2))}\,\mathrm{d}p$ and PolyLog$[n, z]$ is an analytic continuation of $\sum_{k=1}^{\infty} z^k/k^n$. This function can easily be computed by use of *mathematica*.

The main result of this work is that the values of χ (for modestly large $N < 400$) of the critical ensemble due to [11] agree unexpectedly well with the theoretical formula due to [10], equation (10). This agreement is, however, not perfect since the data points lie below equation (10) in the range $3 \leqslant b^{-1} \leqslant 10$ where the function $\chi(b^{-1})$ in (10) has a peculiar curved shape.

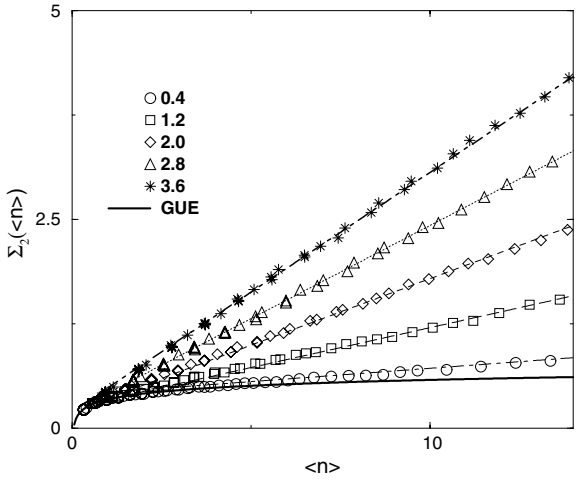

Figure 1. The typical behaviour of the level-number variance $\Sigma_2(\langle n\rangle)$ for $N = 400$ and for $b^{-1} = 0.4$, 1.2, 2.0, 2.8 and 3.6. The lines through the data points correspond to $\chi_0(b) + (\frac{1}{4\pi b})\langle n\rangle$. The value of χ_0 is on average N-independent (see figure 3). The linear behaviour of Σ_2 is in perfect agreement with the theoretical prediction of [18, 24] (see equation (9)). For clarity in the numerical data we skip every third symbol.

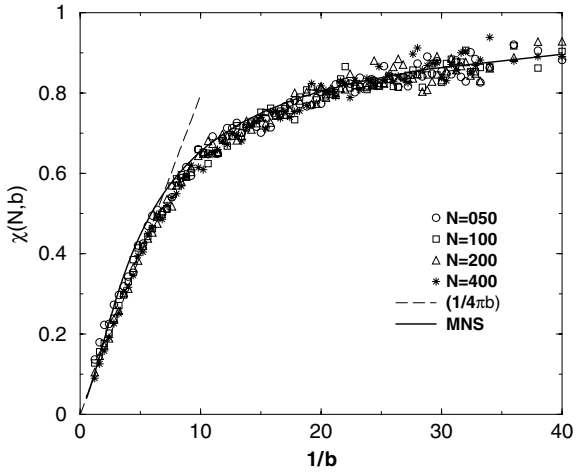

Figure 2. The spectral level compressibility χ as a function of b^{-1} and N. The values of χ were calculated by a fit-polynomial of degree $m = 3$. The solid line is the analytic formula (MNS) due to [10] and the dashed line is equation (9). χ is on average N-independent.

4. Conclusion

We have been able to compute χ as a function of b^{-1} for the GUE critical ensemble. Our results are in agreement with the theoretical formula (10) except for the range $3 \leqslant b^{-1} \leqslant 10$ where the numerical data are below equation (10).

Furthermore, χ_0 and χ are almost N-independent which is a signature of criticality. It is important to note that our simulations exclude the strong N-dependence of χ but they do not exclude the weak $\ln(N)$-dependence of χ which is not trivial to realize numerically. Recently,

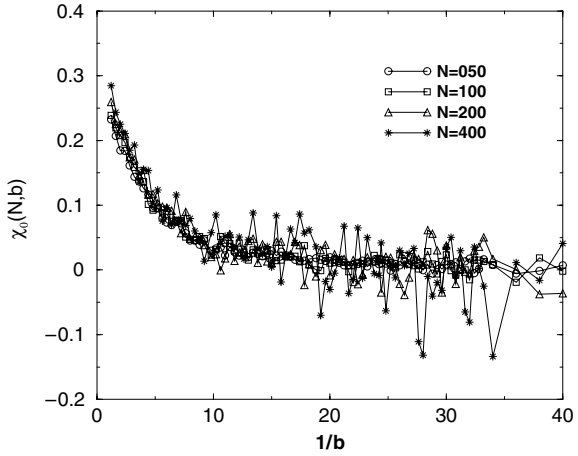

Figure 3. The constant term $\chi_0(N, b)$ of the fit-polynomial with degree $m = 3$ in equation (8). χ_0 decays to zero for values of $b^{-1} \leqslant 10$. It is interesting to note that χ_0 has a finite limit as $b \to 0$. The data here are the same as in figure 2.

it has been shown analytically [27] that in the CrRME the corrections to χ of order b^2 have a logarithmic N-term, i.e. $-b^2 \ln(N)$ in the *crossover* between the GOE and the GUE. In both the GUE and the GOE this logarithmic factor is absent up to order b^2. Its presence is not excluded in higher b corrections, e.g., as $-b^3 \ln(N)$. Such a logarithmic dependence is of fundamental interest. This is because the presence of $\ln(N)$-corrections would signal the *incomplete* criticality of the model. The logarithmic terms may drive the system to a different, stable, set of critical points where the χ may be drastically different from the MNS result. Unfortunately, the accuracy of our simulations is not enough to detect such a weak logarithmic dependence. This point needs further investigation.

We note also that the constant term χ_0 carries important information about CLS since it has a finite limit as $N \to \infty$. As is seen in figure 3, the value of χ_0 as a function of b^{-1} decays to zero with increasing b^{-1} which is (so far unnoticed) an important characteristic feature of CLS.

In summary we have been able to compare via numerical simulations of the GUE critical ensemble [11] and the theoretical formula equation (10). The two results agree but in the region $3 \leqslant b^{-1} \leqslant 10$, the simulations are below the theoretical calculations. However, we emphasize that our results are preliminary and further work is in progress.

Acknowledgments

MLN would like to thank R A Römer for encouraging discussions, funding from the Deutschen Forschungsgemeinschaft through the Sonderforschungsbereich 393 and the ICTP where this work was started.

References

[1] Dyson F J 1962 *J. Math. Phys.* **3** 140
[2] Wigner E P 1951 *Proc. Camb. Phil. Soc.* **47** 790
[3] Mehta M L 1990 *Random Matrices* (Boston MA: Academic)
[4] Haake F 1992 *Quantum Signatures of Chaos* 2nd edn (Berlin: Springer)

[5] Anderson P W 1958 *Phys. Rev.* **109** 1492
[6] Kravtsov V E and Muttalib K A 1997 *Phys. Rev. Lett.* **79** 1913
[7] Janssen M 1998 *Phys. Rep.* **295** 1
[8] Shklovskii B I, Shapiro B, Sears B R, Lambrianidis P and Shore H B 1993 *Phys. Rev.* B **47** 11487
[9] Kravtsov V E, Lerner I V, Altshuler B L and Aronov A G 1994 *Phys. Rev. Lett.* **72** 888
[10] Moshe M, Neuberger H and Shapiro B 1994 *Phys. Rev. Lett.* **73** 1497
[11] Mirlin A D *et al* 1996 *Phys. Rev.* E **54** 3221
[12] Muttalib K A, Chen Y, Ismail M E H and Nicopoulos V N 1993 *Phys. Rev. Lett.* **71** 471
[13] Mirlin A D 2000 *Phys. Rep.* **326** 259
[14] Bogomolny E, Bohigas O, Keating J and Plato M P 1997 *Phys. Rev.* E **55** 6707
[15] Garcia-Garcia A M and Verbaarschot J J M 2000 *Nucl. Phys.* B **586** 668
[16] Blecken C, Chen Y and Muttalib K A 1994 *J. Phys. A: Math. Gen.* **27** L563
[17] Garcia-Garcia A M and Verbaarschot J J M 2002 *Preprint* cond-mat/0204151
[18] Chalker J, Kravtsov V and Lerner I 1996 *Pis'ma Zh. Eksp. Teor. Fiz.* **64** 355 (Engl. Transl. 1996 *Sov. Phys.–JETP Lett.* **64** 386)
[19] Kravtsov V E and Tsvelik A M 2000 *Phys. Rev.* B **62** 9888
[20] Nishigaki S M 1999 *Phys. Rev.* E **59** 2853
[21] Mirlin A D and Evers F 2000 *Phys. Rev.* B **62** 7920
[22] Evers F and Mirlin A D 2000 *Phys. Rev. Lett.* **84** 3690
[23] Canali C M 1996 *Phys. Rev.* B **53** 3713
[24] Altshuler B L, Zharekeshev I K, Kotochigova A A and Shklovskii B I 1988 *Pis'ma Zh. Eksp. Teor. Fiz.* **94** 343 (Engl. Transl. 1988 *Sov. Phys.–JETP Lett.* **67** 625)
[25] Kravtsov V E 1996 *Proc. Correlated Fermions and Transport in Mesoscopic Systems, Moriond Conf. (Les, Arcs, 1996)* (*Preprint* cond-mat/9603166)
[26] Chalker J T, Lerner I V and Smith R A 1996 *Phys. Rev. Lett.* **77** 554
[27] Yevtushenko O M and Kravtsov V E unpublished